Einstein equations for a Finsler-Larange space with canonical N-linear connections

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Abstract: The geometry of Lagrange spaces are applied to the description of classical general relativity and electromagnetic field. This paper, is devoted to study the Einstein equations of Finsler-Lagrange with (α, β)-metrics. By means of the canonical N-metrical connection, the geometry of these spaces can be sufficiently well developed.

Key Words: Finsler Space, Lagrange space, Cartan connection, (α, β) -metrics, Ricci curvature, scalar curvature, torsion tensor.

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1. Introduction

The General theory of relativity is the geometric theory of gravitation published by **Albert Einstein** in 1916. The geometry of Lagrange spaces are applied to the description of classical general relativity, electromegnetic field and Einstein field equations. The theory of Finsler spaces with (α, β)-metrics was introduced by M. Matsumoto [5]. The natural extension of this theory is based on the canonial Cartan nonlinear connection N [1]. The authors H.Shimada and S. Sabau were studied the geometry of Finsler spaces with (α, β)-metrics and on the remarkable classes of Finsler spaces with (α, β)-metric. In [2], the author I. Bucataru have been studied the Finsler space with (α, β)-metrics have nonholonomic frames which are useful for unifying theories in theoretical physics.

The notion of Lorentz nonlinear connection N was introduced by B.T. Hassan, which depends only on the metric $L(\alpha, \beta)$, so the spaces $FL^n = (M, L(\alpha, \beta), N)$ were called the

Finsler-Lagrange spaces with (α,β)-metrics. This thoery has been applied in the study of gravitational and electromagnetic ([9], [10]).

The present paper organized the Euler-Lagrange spaces with (α,β)-metrics and Lorentz equations. Also, Einstein equations for Lagrange space with (α,β)-metrics, in particularly Randers metric by means of canonical N-metrical connection.

2. Preliminaries

In this section, we present some fundamental concepts and facts of Finsler-Lagrange geometry, see([3], [6], [8], [9]).

A. Finsler-Lagrange space with (α,β)-metrics:

Let $F^n = (M, F(x,y))$ be a Finsler space. It has an (α,β)-metric, if the fundamental function $F(x,y)$ can be expressed in the form

$$F(x,y) = \hat{F}(\alpha(x,y), \beta(x,y)),$$

where $\hat{F}$ is a differentiable function of two variables with

$$\alpha^2(x,y) = a_{ij}(x)y^i y^j,$$
$$\beta(x,y) = b_i(x)y^i.$$

Here, α be a pseudo-Riemannian metric on the base manifold M, it gives the gravitational part of $F(x,y)$ and β be the eletromagnetic 1-form on M.

Denoting $L(\alpha(x,y),\beta(x,y)) = \hat{F}(\alpha(x,y),\beta(x,y))$, which follows that $L^n = (M, L)$ is a Lagrange space.

The fundamental metric tensor $g_{ij}(x,y)$ of L^n is

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}.$$

According to [5] g_{ij} can be written as

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}(b_i l_j + b_j l_i) + \rho_{-2} l_i l_j, \tag{2.1}$$

where $b_i = \frac{\partial\beta}{\partial y^i}$, $l_i = a_{ij}y^j = \alpha\frac{\partial\alpha}{\partial y^j}$, ρ, ρ_0, ρ_{-1} and ρ_{-2} are invariants of the space L^n:

$$\rho = \frac{1}{2\alpha}L_\alpha,\ \rho_0 = \frac{1}{2}L_{\beta\beta},\ \rho_{-1} = \frac{1}{2}L_{\alpha\beta},\ \rho_{-2} = \frac{1}{2\alpha^2}(L_{\alpha\alpha} - \frac{1}{\alpha}L_\alpha) \tag{2.2}$$

with

$$\begin{aligned} L_\alpha &= \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2} \\ L_{\beta\beta} &= \frac{\partial^2 L}{\partial \beta^2}, \quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}. \end{aligned}$$

As we known, the Cartan tensor is

$$C_{ijk} = \frac{1}{4}\frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

This tensor is totally symmetric. It can be calculated by means of g_{ij} from (2.1) taking into account the formulae from [12]; one obtains

$$\begin{aligned} 2C_{ijk} &= \sigma_{(i,j,k)}(\rho_{-1}a_{ij}b_k + \rho_{-2}a_{ij}l_k + \frac{1}{3}r_{-1}b_ib_jb_k \\ &\quad + r_{-2}b_ib_jl_k + r_{-3}b_il_jl_k + \frac{1}{3}r_{-4}l_il_jl_k), \end{aligned} \tag{2.3}$$

where $\sigma_{(i,j,k)}$ means the cyclic sum in the indices i, j, k.

B. Variational problem and Lorentz non-linear Connection:

The variational problem can be formulated for differential Lagrangians and can be solved in the case when we consider the parametrized curves, even if the integral of action depends on the parametrization of the considered curve.

Here, we consider the variational problem with end points[4].
Let $L : TM \longrightarrow \mathbb{R}$ be a regular Lagrangian and $c : t \in [0,1] \longrightarrow (x^i(t)) \in U \subset M$ be a regular curve having the image in the domain of the local Chart U of the manifold M. The integral of action of the Lagrangian L on the curve c is given by the functional

$$I(c) = \int_0^1 L(\alpha(x,y), \beta(x,y))dt,$$

which leads to the Euler-Lagrange equations as:

$$E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}(\frac{\partial L}{\partial y^i}) = 0, \quad y^i = \frac{dx^i}{dt}. \tag{2.4}$$

The covector $E_i(L)$ can be expressed as

$$E_i(L) = E_i(\alpha^2) + 2\frac{\rho_{-1}}{\rho}E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial \alpha}{\partial y^i}. \tag{2.5}$$

If c is an extremal curve, i.e., c is a solution of Euler-Lagrange equation (2.4), then along c the energy of a Lagrangian L is

$$E_L = y^i \frac{\partial L}{\partial y^i} - L.$$

Now, let us fix the parametrization of the curve c by a natural parameter $t = s$, with respect to the Riemannian metric $\alpha^2(x, dx/dt)$given by

$$ds^2 = \alpha^2(x, \frac{dx}{dt})dt^2. \tag{2.6}$$

Thus, along the extremal curve c parameterized by arc lengths $t = s$, we have $\alpha^2(x, dx/ds) = 1$ and $d\alpha/ds = 0$, $dL/ds = 0$, which implies that $d\beta/ds = 0$, $dL_\alpha/ds = 0, dL_\beta/ds = 0$. Since $E_i(\beta)$ is given by

$$E_i(\beta) = F_{ij}(x)\frac{dx^j}{ds}, \quad F_{ij} = \frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}. \tag{2.7}$$

Then, the authors R. Miron and B. T. Hassan has obtained the following theorems[10]:

Theorem 2.1. *In the natural parametrization $t = s$ the Euler-Lagrange equations of the Lagrangian $L(\alpha, \beta)$ are given by*

$$E_i(\alpha^2) + 2\frac{\rho_{-1}}{\rho}F_{ij}(x)y^j = 0, \;\; y^i = \frac{\partial x^i}{\partial s}. \tag{2.8}$$

By taking the $\gamma^i_{jk}(x)$ be the Christoffel symbols of the pseudo-Riemannian metric α^2 and

$$\sigma(x, y) = \frac{\rho_{-1}}{\rho}, \quad F^i_j(x) = a^{ih}(x)F_{hj}(x).$$

Therefore the above theorem implies that

Theorem 2.2. *The Euler-Lagrange equation (2.8) are equivalent to the Lorentz equations as*

$$\frac{d^2x^i}{ds2} + \gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = \sigma(x, \frac{dx}{ds})F^i_j(x)\frac{dx^j}{ds}. \tag{2.9}$$

If Euler-Lagrange equations $E_i(L)$=0, then we detemine a canonical semispray S as

$$S = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i},$$

where $2G^i(x, y) = \gamma^i_{jk}(x) - \sigma(x, y)F^i_j(x)y^j$. Them, the integral curve of S are given by the Lorentz equation (2.9).

Now, let us consider the non-linear connection N with the coefficients as

$$N_j^i = \gamma_{jk}^i(x)y^k - \sigma(x)F_j^i(x)$$

Thus, the variation of autoparallel curves of a non-linear connections has been studied in [] and in 2003 some work be progressed.

Since the autoparallel curves of N are given by the Lorentz equation (2.8), we call it is the Lorentz non-linear connection of the metric L and so FL^n is the Finsler- Lagrange (α, β)-metric $L(\alpha, \beta)$ and the Lorentz non-linear connection N. The semispray S associated to N has the coefficients as

$$2G^i = N_j^i y^j. \tag{2.10}$$

Properties of the Lorentz non-linear connection:

(i) The Berwald connection $B\Gamma(N) = (B_{jk}^i(x, y), 0)$ of N has the coefficients

$$B_{jk}^i(x, y) = \gamma_{jk}^i(x) - \dot{\sigma}_k F_j^i(x),$$

where $\dot{\sigma}_k = \frac{\partial \sigma}{\partial y^k}$

(ii) The weak torsion of N is

$$L_{jk}^i = \dot{\sigma}_j F_k^i(x) - \dot{\sigma}_k F_j^i(x).$$

Clearly, if $b_i = grad_i \varphi(x)$, then $L_{jk}^i = 0$.

(iii) The adapted basis are

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

(iv) The integrability tensor

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j},$$

of N is

$$R_{jk}^i = y^h \rho_{hjk}^i(x) + \sigma_j F_k^i - \sigma_k F_j^i - \sigma(F_{j|k}^i - F_{k|j}^i),$$

where $\sigma_j = \frac{\partial \sigma}{\partial x^j}$ and '|' is the covariant derivative with respect to the Levi-Civita connection of α^2 and $\rho_{hik}^i(x)$ is the curvature tensor of the Levi-Civita connection.

(v) The Lorentz non-linear connection N is integrable if and only if the d-tensor of integrability R^i_{jk} vanishes.

(vi) The dual basis $(dx^i, \delta y^i)$ of $(\delta/\delta x^i, \partial/\partial y^i)$ is determined by

$$\begin{aligned}\delta y^i &= dy^i + N^i_j dx^j, \\ &= dy^i + \gamma^i_{jk} y^k dx^j - \sigma F^i_j dx^j, \\ &= \delta y^i - \sigma F^i_j dx^j.\end{aligned}$$

(vii) The autoparallel curves of Lorentz non-linear connection are given by the system of differential equations:

$$\frac{\delta y^i}{dt}, \quad y^i = \frac{dx^i}{dt}.$$

(viii) In the parametrization S with $\alpha^2(x, dx/ds) = 1$, the property (vii) are the Lorentz equation (2.8).

(ix) The exterior differential of 1-forms δy^i of the form

$$d\delta y^i = \frac{1}{2} R^i_{jk} dx^k \wedge dx^j + B^i_{jk} \delta y^k \wedge dx^j.$$

C. Canonical N-metrical connection:

The metric N-linear connection is called the canonical N-linear connection or the Cartan connection of the Lagrange space. The space $FL^n = (M, L(\alpha, \beta), N)$ has a canonical N-linear connection $C\Gamma(N)$ with the coefficients (L^i_{jk}, C^i_k) given by the generalized Chistoffel symbols ([6], [8]):

$$\left.\begin{aligned} L^i_{jk} &= \tfrac{1}{2} g^{is} \left(\tfrac{\delta g_{sk}}{\delta x^j} + \tfrac{\delta g_{js}}{\delta x^k} - \tfrac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} &= \tfrac{1}{2} g^{is} \left(\tfrac{\partial g_{sk}}{\partial y^j} + \tfrac{\partial g_{js}}{\partial y^k} - \tfrac{\partial g_{jk}}{\partial y^s} \right) \end{aligned}\right\} \tag{2.11}$$

The 1-form connection $C\Gamma(N)$ is

$$\omega^i_j = L^i_{jk} dx^k + C^i_{jk} \delta y^k.$$

By the property (ix), the structure equations of $C\Gamma(N)$ expressed in the following theorem[3]:

Theorem 2.3. *The structure equations of the canonical N-linear metrical connection $C\Gamma(N)$ of the space FL^n are as follows*

$$d(dx^i) - dx^k \wedge \omega^i_k = -{}^1\Omega^i,$$
$$d(\delta y^i) - \delta y^k \wedge \omega^i_k = -{}^2\Omega^i,$$
$$d\omega^i_k - \omega^k_j \wedge \omega^i_k = -\Omega^i_j,$$

where ${}^1\Omega$, ${}^2\Omega$ are the 2-forms of torsion.

Here

$$\begin{aligned} {}^1\Omega^i &= C^i_{jk} dx^j \wedge \delta y^k, \\ {}^2\Omega^i &= \frac{1}{2} R^i_{jk} dx^j \wedge dx^k + P^i_{jk} dx^j \wedge \delta y^k \end{aligned}$$

and Ω is the 2-form of curvature.

$$\Omega^i_j = \frac{1}{2} R^i_{jkh} dx^k \wedge dx^h + P^i_{jkh} dx^k \wedge \delta y^h + \frac{1}{2} S^i_{jkh} \delta y^k \wedge \delta y^h,$$

where R^i_{jk} is tensor given in property (iv), $P^i_{jk} = B^i_{jk} - L^i_{jk}$ and R^i_{jkh}, P^i_{jkh}, S^i_{jkh} are the curvature tensor of $C\Gamma(N)$ [6].

In this study, we use the metric N-linear connection $D\Gamma(N) = (\bar{L}^i_{jk}, \bar{C}^i_{jk})$and has a given d-tensor of torsion $\bar{\mathrm{T}}^i_{jk}$ and $\bar{S}^i_{jk}$ as follows:

$$\left.\begin{aligned} \bar{L}^i_{jk} &= L^i_{jk} + \tfrac{1}{2} g^{ih} \left(g_{jr} \bar{\mathrm{T}}^r_{kh} + g_{jh} \bar{\mathrm{T}}^r_{jh} - g_{hr} \bar{\mathrm{T}}^r_{kj} \right), \\ \bar{C}^i_{jk} &= C^i_{jk} + \tfrac{1}{2} g^{ih} \left(g_{jr} S^r_{kh} + g_{jh} S^r_{jh} - g_{hr} S^r_{kj} \right), \end{aligned}\right\} \tag{2.12}$$

where (L^i_{jk}, C^i_{jk}) are the local coefficients of the canonical metric N-linear connection $C\Gamma(N)$ and $\bar{\mathrm{T}}^i_{jk}$, $\bar{S}^i_{jk}$ simply by $(\mathrm{T}^i_{jk}, S^i_{jk})$.

D. Einstein Equations on TM:

Let TM be endowed with a non-linear connection N, an h-v metric structure G and a metrical N-connection $D\Gamma(N)$ with a priori given torsions $(\mathrm{T}^i_{jk}, S^i_{jk})$.

Given an h-v metric G on TM becomes a pseudo-Riemannian manifold of dimension $2n$. The Einstein equations written for the connection $D\Gamma(N)$ on TM as:

$$Ric(D) - \frac{1}{2}Sc(D)G = k\mathrm{T}, \tag{2.13}$$

where $Ric(D)$ is the Ricci tensor field and $Sc(D)$ is the scalar curvature of $D\Gamma(N)$, k is constant and T is the energy-momentum tensor field.
In local coordinates, the authors R. Miron and I. Bucataru were stated as[7]:

Theorem 2.4. *The Einstein equations of the Lagrange space $L^n = (M, L)$ corresponding to the metric N-linear connection $D\Gamma(N) = (L^i_{jk}, C^i_{jk})$ with the coefficients (2.12) have the following form*

$$R_{ij} = \frac{1}{2}(R+S)g_{ij} = k\mathrm{T}_{ij},$$
$$S_{ij} = \frac{1}{2}(R+S)g_{ij} = k\mathrm{T}_{(i)(j)},$$
$${}^1P^i_j = k\mathrm{T}_{(i)j},\ \ {}^2P^i_j = -k\mathrm{T}_{i(j)},$$

where T_{ij}, $\mathrm{T}_{(i)(j)}$, $\mathrm{T}_{i(j)}$ are d-tensor fields.

3. The Notion of Randers metric

The preliminaries theories has a remarkable particular case, that is based on the Randers metric.

$$F(x,y) = \alpha(x,y) + \beta(x,y). \tag{3.1}$$

The Lagrange space $L^n = (M, L)$ with

$$L(\alpha(x,y), \beta(x,y)) = \hat{F}^2(\alpha(x,y), \beta(x,y)) = (\alpha+\beta)^2. \tag{3.2}$$

The invariants (2.2) of Randers metric are given by

$$\rho = \frac{\alpha+\beta}{\alpha},\ \rho_0 = 1,\ \rho_{-1} = \frac{1}{\alpha},\ \rho_{-2} = -\frac{\beta}{\alpha^3}.$$

Using the formula (2.1), we obtain the fundamental metric tensor g_{ij},

$$g_{ij} = \frac{\alpha+\beta}{\alpha}a_{ij} + b_ib_j + \frac{1}{\alpha}(b_il_j + b_jl_i) - \frac{\beta}{\alpha^3}l_il_j. \tag{3.3}$$

Its contravariant counterpart g^{ij} as

$$g^{ij} = \frac{1}{\rho}\left(a^{ij} - \frac{(b^i l^j + b^j l^i)}{\alpha+\beta} + \frac{b^2}{(\alpha+\beta)^2} y^i y^j\right). \tag{3.4}$$

And we know g_{ij} is positively defined if $b^2 < 1$. The Cartan tensor C_{ijk} (2.3) given by

$$C_{ijk} = \sigma_{(i,j,k)} \frac{1}{2}\left(\frac{1}{\alpha} a_{ij} b_k - \frac{\beta}{\alpha^3} a_{ij} l_k - \frac{1}{\alpha^3} b_i l_j l_k + \frac{\beta}{\alpha^4} l_i l_j l_k\right). \tag{3.5}$$

Clearly, we see that $C_{ijk} \neq 0$. Thus, we have

Theorem 3.5. *The Cartan tensor C_{ijk} of Randers metric is non zero (different from zero).*

Moreover, the Raders metric is not reducible to a Riemannian metric. For this metric (3.2), the Euler-Lagrange equation in the natural parametrization given by

$$E_i(\alpha^2) + 2\sigma F_{ij} y^j = 0, \;\; y^i = \frac{dx^i}{ds}, \tag{3.6}$$

where $\sigma = \frac{\rho_{-1}}{\rho} = \frac{1}{2}\frac{L_\beta}{\rho} = \alpha$.

From theorem 2.2 , we have the result

Theorem 3.6. *The Euler-Lagrange equations of Randers metric $L = (\alpha+\beta)^2$ is the natural parametrization $\alpha(x, dx/ds) = 1$ are given by the Lorentz equation*

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = \alpha F^i_j(x)\frac{dx^j}{ds} \tag{3.7}$$

Thus, the coefficients of the canonical semispray and non-linear connection N as:

$$2G^i(x,y) = \gamma^i_{jk}(x) y^j y^k - \alpha(x,y) F^i_j(x) y^j,$$

$$N^i_j = \gamma^i_{jk}(x) y^k - \alpha F^i_j(x).$$

The weak torsion of N is $L^i_{jk} = 0$ and the metric N-linear connection $D\Gamma(N) = (\bar{L}^i_{jk}, \bar{C}^i_{jk})$ is given in (2.12) coincide with those of the Cartan connection. Moreover, taking into account that, with respect to the canonical N-linear connection N, we have $\frac{\delta F}{\delta x^i} = 0$.

The torsion tensor of $D\Gamma(N)$ are

$$\mathrm{T}^i_{jk} = 0, \;\; R^i_{jk}, \;\; C^i_{jk}, \;\; P^i_{jk} = N^i_{jk} - L^i_{kj}, \;\; S^i_{jk} = 0. \tag{3.8}$$

In the following and using the properties form [1], we get

$$P^i_{jk} y^k = 0, \;\; P^i_{jk} y^j = 0. \tag{3.9}$$

4. Einstein Equations of Lagrange space with Randers metric

In this section, we find the Einstein equations for Lagrange space with Randers metric.

Now, we shall express equation (2.13) in the basis $(\delta/\delta x^i, \partial/\partial y^i)$, i.e., adapted to the decomposition of T_uTM, $u \in TM$ into horizontal and vertical subspaces. Recall that such decomposition is produced by the N-linear connection derived form L.

Regarding this, we set $(X_\alpha) = (X_i, X_{(i)})$, where $X_i = \delta/\delta x^i$ and $X_{(i)} = \partial/\partial y^i$. The indices i will run from 1 to $2n$ and (i) will run from $n+1$ to $2n$. The local vector fields (X_a) provides a nonholonomic basis given by

$$[X_b, X_c] = W^a_{bc} X_a,$$

which satisfies the following Vranceanu indentities [13]

$$\sum_{(abc)} [X_a(W^d_{bc}) + W^e abW^d_{ce}] = 0.$$

Let $D_{X_c} X_b = \Gamma^a_{bc} X_a$. Then the basis (X_a) the torsion T of the N-linear connection D has the components

$$\mathrm{T}^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} + W^a_{bc}.$$

In the basis (X_a) the curvature R of the N-linear connection D has the components

$$R^a_{bcd} = X_d \Gamma^a_{bc} - X_c \Gamma^a_{bd} + \Gamma^e_{bc} \Gamma^a_{ed} - \Gamma^e_{bd} \Gamma^a_{ec} + \Gamma^a_{be} \Gamma^e_{cd}.$$

The torsion and curvature components given by

$$\mathrm{T}(X_c, X_b) = \mathrm{T}^a_{bc} X_a, \quad R(X_d, X_c) X_b = R^a_{bcd} X_a.$$

In the adapted basis (X_a) the Bianchi identities of D of the form:

$$\sum_{abc} (D_a R^e_{dbc} + R^e_{dav} \mathrm{T}^v_{bc}) = 0,$$

$$\sum_{abc} (D_a \mathrm{T}^d_{bc} + \mathrm{T}^e_{ab} \mathrm{T}^d_{ec} - R^d_{abc}) = 0,$$

where $D_a = DX_a$.

If in these equations the components with respect to $X_i = \delta/\delta x^i$ and $X_{(i)} = \partial/\partial y^i$ are separated, it comes out that among the coefficients Γ^a_{bc}, we have

$$\Gamma^i_{jk} = L^i_{jk}, \quad \Gamma^{(i)}_{(j)(k)} = C^i_{jk}.$$

This is advantage created by the choice of the basis (X_a) as well as by the fact that D is an N-linear connection.

The set of components T^a_{bc} of the torsion field T splits into following:

$$\left.\begin{array}{l}\mathbb{T}^i_{jk} = \mathrm{T}^i_{jk},\ \mathrm{T}^i_{(j)k} = -C^i_{jk},\ \mathbb{T}^i_{j(k)} = -C^i_{jk},\ \mathbb{T}^{(j)(k)} = 0,\\ \mathbb{T}^{(i)}_{jk} = R^i_{jk},\ \mathbb{T}^{(i)}_{(j)k} = -P^i_{kj},\ \mathbb{T}^{(i)}_{j(k)} = P^i_{jk},\ \mathbb{T}^{(i)}_{(j)(k)} = 0.\end{array}\right\} \tag{4.1}$$

with respect to the basis (X_a), the Ricci tensor field of the N-linear connection $D\Gamma(N)$ has the components

$$\mathbb{R}_{ij} = Rij,\ \mathbb{R}_{(i)j} =^1 P_{ij},\ \mathbb{R}_{i(j)} = -^2P_{ij},\ \mathbb{R}_{(i)(j)} = S_{ij}.$$

By the pseudo-Riemannian metric G has the components G_{ab} given by

$$G_{ij} = g_{ij},\ G_{i(j)} = 0,\ G_{(i)j} = 0,\ G_{(i)(j)} = g_{ij},$$
$$G^{ij} = g^{ij},\ G^{i(j)} = 0,\ G^{(i)j} = 0,\ G^{(i)(j)} = g^{ij},$$

where g_{ij} and g^{ij} are given in (3.3) and (3.4) respectively.

Thus, the tensor field $\mathbb{R}^a_b = G^{ac}R_{cb}$ and the scalar curvature $Sc(D)$ have in the frame X_a the components are

$$\mathbb{R}^i_j = R^i_j,\ \mathbb{R}^{(i)}_j =^1 P^i_j,\ \mathbb{R}^i_{(j)} =^2 P^i_j,\ \mathbb{R}^{(i)}_{(j)} = S^i_j,\ Sc(D) = R + S,$$

where $R = g^{ij}R_{ij}$ and $S = g^{ij}S_{ij}$.

Theorem 4.7. *The Einstein equations of the Largrange space with Randers metric corresponding to the metric N-linear connection $D\Gamma(N) = (L^i_{jk}, C^i_k)$ have the following form*

$$\left.\begin{array}{l}R_{ij} = \frac{1}{2}(R+S)g_{ij},\\ {}^1P^i_j = 0,\quad {}^2P^i_j = 0,\\ S_{ij} = \frac{1}{2}(R+S)g_{ij},\end{array}\right\} \tag{4.2}$$

where g_{ij} given in (3.3).

Proof. Making use of the formulae (4.2), one can shows that from theorem and corresponding d-tensor fields in (2.12), (3.8), (3.9) and (4.1) are equivalent to get (4.2). □

In vaccum, which corresponds to the case $\mathrm{T}_{ij} = 0$, if we multiply this with $G^{ij} = g^{ij}$ the equation (2.13) of Randers metric can be written in the form

$$R_{ij} - \frac{1}{2}Sc(D)G_{ij} = 0, \text{ or } R_{ij} - \frac{1}{2}Sc(D)g_{ij} = 0, \tag{4.3}$$

which implies that $Sc(D) - nSc(D) = o$. Hence, $Sc(D) = 0$ for $n > 1$. Thus,the equation (**??**) takes the form $R_{ij} = 0$ and immediately we obtain the following result.

Lemma 4.1. *For the vaccum state, the Einstein equations of the Lagrange space with Randers metric corresponding to the metric connection $D\Gamma(N) = (L^i_{jk}, C^i_k)$ are as follows*

$$R_{ij} = 0, \ S_{ij} = 0, \ {}^1P^i_j = 0, \ {}^2P^i_j = 0. \tag{4.4}$$

References

[1] D. Bao, S. S. Chern and Z. Shen, *An Introdction to Riemannian-Finsler geometry*, Springer, Berlin, 2000.

[2] I. Bucataru, *Nonholonomic frames for Finsler sapces with (α, β)-metrics*, Proceedings of the Conference on Finsler and Lagrange geometries, lasi, August 2001, Kluwer Acad. Publ. 69-78(2003).

[3] I. Bucataru, and R. Miron., *Finsler-Lagrange Geometry. Applications to Dynamical system*, Editura Academiei Romane, Bucuresti, 2007.

[4] Gelfand M. F., Fomin S. V., *Calculs of variations*, Dover Publications (2000).

[5] M. Matsumoto, *Foundation of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Otsu 1986.

[6] R. Miron and M. Anastasiei, *The Geometry of Lagrange spaces*, Theory and applications, Kluwer Acad. Publ. FTPH 59, 1994.

[7] R. Miron and M. Anastasiei, *Vector Bundles, Lagrange spaces, Theory and apllications*, (in Romanian), Ed. Acad., Bucharest, 1987.

[8] R. Miron and D. Hrimiuc, H. Shimada and S. Sabau, *The Geometry of Hamilton and Lagrange spaces*, Kluwer Acad, Publ. FTPH 118, 2001.

[9] R. Miron and B.T. Hassan, *Variational problem in Finsler spaces with (α, β)-metric*, Algebras, Groups and Geometries, Hadronic Press, vol. 20, p.285-300, 2003.

[10] R. Miron and B.T. Hassan, *Gravitation and Eletromagnetism in Finsler-Lagrange spaces with (α, β)-metric.*

[11] S.K. Narasimhamurthy, Mallikarjun Y.Kumbar and Kavyashree A.R., *Finslerian Nonholonomic Frames for Matsumoto (α, β)-metrics*, International Journal of Mathematics and Statistics Invention, E-ISSN 2321-4767 Volume 2, Issue 3,pp 73-77, March(2014).

[12] H. Shimada and S. Sabau, *Remarkable classes of (α, β)-metric sapce*, Rep. Math. Phy. 47(2001), 31-48.

[13] Vranceanu. G., *Lecons de geometrie differentielle*, Editura Academiei Romane, Bucuresti, 1957.